Lecture 9 -- additional remarks on algebraic stacks

Thursday, September 29, 2016 2:03 PM

Last time: example of stack X/G

Another way to think about this:

Groupoid scheme $X_\bullet \rightsquigarrow$ presheaf of groupoids on Sch/k (the naive 1-category)

$\rightsquigarrow$ Fibered category over Sch/k

$$b: \mathcal{F} \to Sch$$

call it $\underline{X_\bullet}$: $\mathcal{F}$ has objects $(U, \xi \in X_0(U))$, maps

$$\begin{tikzcd} \xi' \arrow[r, "\gamma"] \arrow[d, maps to] & \xi \arrow[d, maps to] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$

is a $\gamma \in X_1(U)$ with $t(\gamma) = \xi'$, $s(\gamma) = f^*(\xi)$

For any category fibered in groupoids, $\exists$ a stackification

$$\mathcal{F} \to \mathcal{F}^a$$

the map $\mathcal{F} \to \mathcal{F}^a$ is universal for $\underline{\text{maps}}$ to stacks

$\hookrightarrow$ maps in the 2-category of fibered categories are base preserving functors (strictly commute w/

categories are base preserving functors (strictly commute w/ projection to Sch)

Some key remarks:

1) 2-fiber products of categories

Tag 003O From <http://stacks.math.columbia.edu/tag/003O>

Example 4.30.3. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. We define a category $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ as follows:

(1) an object of $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is a triple (A, B, f), where $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and $f : F(A) \to G(B)$ is an isomorphism in $\mathcal{C}$,
(2) a morphism $(A, B, f) \to (A', B', f')$ is given by a pair (a, b), where $a : A \to A'$ is a morphism in $\mathcal{A}$, and $b : B \to B'$ is a morphism in $\mathcal{B}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "f"] \arrow[d, "F(a)"] & G(B) \arrow[d, "G(b)"] \\ F(A') \arrow[r, "f'"] & G(B') \end{tikzcd}$$

is commutative.

Moreover, we define functors $p : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{A}$ and $q : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{B}$ by setting

$$p(A, B, f) = A, \quad q(A, B, f) = B,$$

in other words, these are the forgetful functors. We define a transformation of functors $\psi : F \circ p \to G \circ q$. On the object $\xi = (A, B, f)$ it is given by $\psi_\xi = f : F(p(\xi)) = F(A) \to G(B) = G(q(\xi))$.

If $\mathcal{X}_0, \mathcal{X}_1 \to \mathcal{Y}$ are base preserving maps of stacks, then $\mathcal{X}_0 \times_{\mathcal{Y}} \mathcal{X}_1$ is still a stack, w/ commutative diagram

$$\begin{tikzcd} \mathcal{X}_0 \times_{\mathcal{Y}} \mathcal{X}_1 \arrow[r] \arrow[d] & \mathcal{X}_1 \arrow[d] \arrow[dl, phantom, "\swarrow \eta"] \\ \mathcal{X}_0 \arrow[r] & \mathcal{Y} \end{tikzcd}$$

<u>universal</u>

$$\begin{tikzcd} {} \arrow[d, "F"] & {} \arrow[d] \arrow[l, phantom, "\swarrow \eta"] \\ \mathcal{X}_0 \arrow[r] & \mathcal{Y} \end{tikzcd}$$

universal

3) 2-yoneda lemma (see Vistoli)

2-YONEDA LEMMA. *The two functors above define an equivalence of categories*

$$\mathrm{Hom}_{\mathcal{C}}((\mathcal{C}/X), \mathcal{F}) \simeq \mathcal{F}(X).$$

implication: can think of schemes as fibered categories too, no loss of data.

Rem: Can also restrict to affines schemes Alg/k

Ex: (of many things at once) (skip?)

Let $X_\bullet$ be a groupoid scheme.
- regard X_0, X_1, X_2 as fibered categories
- let $\mathcal{X}$ be cat fibered in groupoids assoc. to $X_\bullet$
- Let $\mathcal{X} \to \mathcal{X}^a$ be the stackification

Then consider...

$$\begin{tikzcd} & & & \mathrm{QCoh} \arrow[d] \\ \mathrm{Colim}(X_\bullet) \cong \mathcal{X} \arrow[r] \arrow[urrr, dashed] & \mathcal{X}^a \arrow[rr] \arrow[urr, dashed] & & \mathrm{Sch}/k \end{tikzcd}$$

Categories of sections are all equivalent, so what we defined as $\mathrm{QCoh}(X_\bullet)$ is the category of sections.

equivalent, so what we defined as $QCoh(X_\bullet)$ is the category of sections.

Thm. TFAE for a cat fibered in groupoids

1) $\mathcal{X} \cong (\underline{X}_\bullet)^a$ for a smooth groupoid
2) $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable* and $\exists$ smooth surjection $U \to \mathcal{X}$ from a scheme
3) $\exists$ representable* smooth surjection $U \to \mathcal{X}$

Furthermore, these are equiv. to the same, but with fppf replacing smooth. Such an $\mathcal{X}$ is an alg. stack

Rem: Not true that fppf maps admit etale local sections, but there's a construction, for any fppf map $U \to \mathcal{X}$, of a $U' \to U$ s.t. composition $U' \to \mathcal{X}$ is smooth surjection.

Tag 06DC From <http://stacks.math.columbia.edu/tag/06DC>

Rem: discuss correspondence betw. groupoid and stack

Example 1: $X \to B$ projective or proper morphism

Tag 0372

Tag 0372

78.16. The Picard stack

Let S be a scheme. Let $\pi : X \to B$ be a morphism of algebraic spaces over S. We define a category $Pic_{X/B}$ as follows:

1. An object is a triple $(U, b, \mathcal{L})$, where
   - (a) U is an object of $(Sch/S)_{fppf}$,
   - (b) $b : U \to B$ is a morphism over S, and
   - (c) $\mathcal{L}$ is in invertible sheaf on the base change $X_U = U \times_{b,B} X$.
2. A morphism $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ is given by a morphism of schemes $f : U \to U'$ over B and an isomorphism $g : f^*\mathcal{L}' \to \mathcal{L}$.

The composition of $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ with $(f', g') : (U', b', \mathcal{L}') \to (U'', b'', \mathcal{L}'')$ is given by $(f \circ f', g \circ f^*(g'))$. Thus we get a category $Pic_{X/B}$ and

$$p : Pic_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{L}) \longmapsto U$$

Is algebraic, slight modification: Picard scheme, stack of coherent sheaves

Ex2: Stack of flat families of curves w/

a) geometrically reduced & connected fibers of arithmetic genus $h^1(\mathcal{O}_X) = 1$

↳ algebraic

b) Flat family of curves $\mathcal{C} \to B$ + relatively ample bundle

Separation axiom: we usually work with geometric stacks, meaning

$\mathcal{X} \to \mathcal{X} \times \mathcal{X}$

with geometric stacks, meaning

$$\mathfrak{X} \longrightarrow \mathfrak{X} \times \mathfrak{X}$$

is affine.

$$\begin{tikzcd} \underline{\mathrm{Isom}}_U(\xi_1, \xi_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{X} \arrow[d] \\ U \arrow[r, "(\xi_1, \xi_2)"'] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

is this affine? yes in Ex 1, Ex 2a

No in Ex 2b: look at smooth elliptic curve degenerating to nodal elliptic, so behavior of automorphism groups is weird

Rem: maps of alg. stacks are repres. iff the maps $\mathfrak{X}(U) \to \mathcal{Y}(U)$ are faithful

Given homom. $\varphi: G \to H$ and an equivariant map $f: X \to Y$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \circlearrowleft & & \circlearrowleft \\ G & \xrightarrow{\varphi} & H \end{array}$$

$\Longrightarrow$ map of groupoids

$$\begin{tikzcd} G \times X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \arrow[d] \\ H \times Y \arrow[r, shift left] \arrow[r, shift right] & Y \end{tikzcd}$$

$\Longrightarrow$ map of stacks

$$\begin{tikzcd} X/G \arrow[r] & Y/H \\ X \times H/G \arrow[u] \arrow[r] \arrow[ur, phantom, "\llcorner"] & Y \arrow[u] \end{tikzcd}$$

so map is representable iff

so map is representable

iff

G acts freely on $X \times H$ $\rightsquigarrow$ $X \times H/G \longrightarrow Y$

e.g. if $G \hookrightarrow H$, then

$$X/G \underset{\substack{\uparrow \\ \text{Shapiro's} \\ \text{lemma}}}{\cong} (X \times_G H)/H$$

And the map is modeled by the H-equivariant map $X \times_G H \longrightarrow Y$

$$(x,h) \longmapsto h \cdot f(x)$$

<u>Cor:</u> X/G is geometric for X separated

<u>Pf:</u> $X/G \cong X \times_G (G \times G)/_{G \times G} \cong X \times \{1\} \times G/_{G \times G}$

where $(g_1, g_2) \cdot (h, 1, x) = (g_1 h, g_2, x)$

$$\sim (g_1 h g_2^{-1}, 1, g_2 x)$$

diagonal is presented by $G \times G$-equivariant map

$$G \times X \longrightarrow X \times X$$
$$(h,x) \longmapsto (x, hx)$$

preimage over any $\mathrm{Spec}(R) \longrightarrow X \times X$ is

$$\begin{tikzcd} G\times \mathrm{Spec}(R) \arrow[r, "\psi"] & X \times \mathrm{Spec}(R) \\ S \arrow[u, hook] \arrow[r, "i"] & \mathrm{Spec}(R) \arrow[u, "\Gamma"'] \end{tikzcd}$$

closed imm.

affine!

$\boxtimes$

<u>Rem:</u> Geometric stack presented by groupoid in <u>affines</u>

One other key result: Totaro, "The resolution property for schemes and stacks"

Theorem 1.1 *Let X be a normal noetherian algebraic stack (over $\mathbf{Z}$) whose stabilizer groups at closed points of X are affine. The following are equivalent.*

(1) X has the resolution property: every coherent sheaf on X is a quotient of a vector bundle on X.

(2) X is isomorphic to the quotient stack of some quasi-affine scheme by an action of the group $GL(n)$ for some n.

For X of finite type over a field k, these are also equivalent to:

(3) X is isomorphic to the quotient stack of some affine scheme over k by an action of an affine group scheme of finite type over k.